

The simplicity of Kac modules for the quantum superalgebra $U_q(gl(m, n))$

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1 Introduction

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be the general linear Lie superalgebra over the complex number field \mathbb{C} . The quantum superalgebra $U_q(\mathfrak{g})$ in the present paper was defined by R. Zhang [12]. The Kac module $K(M)$ is the $U_q(\mathfrak{g})$ -module induced from a simple $U_q(\mathfrak{g}_{\bar{0}})$ -module M . Assume M is a weighted $U_q(\mathfrak{g}_{\bar{0}})$ -module which is generated by a primitive vector of weight λ . Then λ is called typical if $K(M)$ is simple. The typical weights in both generic case and the case where q is a primitive root of unity were first studied in [12]. Also in [5], a sufficient condition for the typicality is given in generic case.

One of the main goals of the present paper is to determine the typical weights. We prove that in the case where $K(M)$ is weighted, the typical weights are determined by a polynomial. Then we determine the polynomial using the method provided by [11]. Let us note that our polynomial coincides with one given in [12], despite the fact that the order of the product for the elements $F_{ij}((i, j) \in \mathcal{I}_1)$ used in [12] to define the polynomial is completely different from ours.

The paper is organized as follows. Sec. 3 is the preliminaries. In Sec. 4, we give some identities in $U_q(\mathfrak{g})$. In Sec. 5 we discuss the simplicity of the Kac modules, which is determined by a polynomial. The polynomial is determined in Sec. 6. In Sec. 7, we study the simple modules in the case where q is a l th root of unity. We prove that, under certain conditions, the algebras $u_{\eta, \chi}(\mathfrak{g}_{\bar{0}})$ and $u_{\eta, \chi}$ are Morita equivalent.

2 Notations

Throughout the paper we use the following notation.

$[1, m+n)$	$= \{1, 2, \dots, m+n-1\}.$
$[1, m+n]$	$= \{1, 2, \dots, m+n\}.$
A^{m+n}	the set of all $m+n$ -tuples $z = (z_1 \dots z_{m+n})$ with $z_i \in A$ for all $i = 1, \dots, m+n$
\mathcal{I}_0	$= \{(i, j) 1 \leq i < j \leq m \text{ or } m+1 \leq i < j \leq m+n\}$
\mathcal{I}_1	$= \{(i, j) 1 \leq i \leq m < j \leq m+n\}$
\mathcal{I}	$= \mathcal{I}_0 \cup \mathcal{I}_1$
A^B or $B = \mathcal{I}_1$	the set of all tuples $\psi = (\psi_{ij})_{(i,j) \in B}$ with $\psi_{ij} \in A$, where $B = \mathcal{I}_0$
\mathcal{A}	$= \mathbb{C}[q]$ where q is an indeterminate
$h(V)$ $V = V_{\bar{0}} \oplus V_{\bar{1}}$	the set of all homogeneous elements in a \mathbb{Z}_2 -graded vector space
\bar{x}	the parity of the homogeneous element $x \in V = V_{\bar{0}} \oplus V_{\bar{1}}$.
$U(L)$ L .	the universal enveloping superalgebra for the Lie superalgebra

3 The quantum deformation of $gl(m, n)$

The general linear Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ has the standard basis([7]) e_{ij} , $1 \leq i, j \leq m+n$. We denote e_{ji} with $i < j$ also by f_{ij} . Then we get $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_1$, where

$$\mathfrak{g}_1 = \langle e_{ij} | (i, j) \in \mathcal{I}_1 \rangle \quad \mathfrak{g}_{-1} = \langle f_{ij} | (i, j) \in \mathcal{I}_1 \rangle.$$

The parity of the basis elements is given by

$$\bar{e}_{ij} = \bar{f}_{ij} = \begin{cases} \bar{0}, & \text{if } (i, j) \in \mathcal{I}_0 \text{ or } i = j \\ \bar{1}, & \text{if } (i, j) \in \mathcal{I}_1. \end{cases}$$

Let $H = \langle e_{ii} | 1 \leq i \leq m+n \rangle$. Then the set of positive roots of \mathfrak{g} relative to H is $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$, where

$$\Phi_0^+ = \{\epsilon_i - \epsilon_j | (i, j) \in \mathcal{I}_0\}, \quad \Phi_1^+ = \{\epsilon_i - \epsilon_j | (i, j) \in \mathcal{I}_1\}.$$

Let $\Lambda = \mathbb{Z}\epsilon_1 + \dots + \mathbb{Z}\epsilon_{m+n} \subseteq H^*$. There is a symmetric bilinear form defined on Λ as follows([10]):

$$(\epsilon_i, \epsilon_j) = \begin{cases} \delta_{ij}, & \text{if } i < m \\ -\delta_{ij}, & \text{if } i > m. \end{cases}$$

Let q be an indeterminate over \mathbb{C} . Then the quantum supergroup $U_q(\mathfrak{g})$ (see [12, p.1237]) is defined as the $\mathbb{C}(q)$ -superalgebra with the generators $K_j, K_j^{-1}, E_{i,i+1}, F_{i,i+1}, i \in [1, m+n]$, and relations

$$(R1) \quad K_i K_j = K_j K_i, K_i K_i^{-1} = 1,$$

$$(R2) \quad K_i E_{j,j+1} K_i^{-1} = q_i^{(\delta_{ij} - \delta_{i,j+1})} E_{j,j+1}, \quad K_i F_{j,j+1} K_i^{-1} = q_i^{-(\delta_{ij} - \delta_{i,j+1})} F_{j,j+1},$$

$$(R3) \quad [E_{i,i+1}, F_{j,j+1}] = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q_i - q_i^{-1}},$$

$$(R4) \quad E_{m,m+1}^2 = F_{m,m+1}^2 = 0,$$

$$(R5) \quad E_{i,i+1} E_{j,j+1} = E_{j,j+1} E_{i,i+1}, \quad F_{i,i+1} F_{j,j+1} = F_{j,j+1} F_{i,i+1}, |i-j| > 1,$$

$$(R6) \quad E_{i,i+1}^2 E_{j,j+1} - (q+q^{-1}) E_{i,i+1} E_{j,j+1} E_{i,i+1} + E_{j,j+1} E_{i,i+1}^2 = 0 \quad (|i-j| = 1, i \neq m),$$

$$(R7) \quad F_{i,i+1}^2 F_{j,j+1} - (q+q^{-1}) F_{i,i+1} F_{j,j+1} F_{i,i+1} + F_{j,j+1} F_{i,i+1}^2 = 0 \quad (|i-j| = 1, i \neq m),$$

$$(R8) \quad [E_{m-1,m+2}, E_{m,m+1}] = [F_{m-1,m+2}, F_{m,m+1}] = 0,$$

where

$$q_i = \begin{cases} q, & \text{if } i \leq m \\ q^{-1}, & \text{if } i > m. \end{cases}$$

Most often, we shall use E_{α_i} (resp. $F_{\alpha_i}; K_{\alpha_i}$) to denote $E_{i,i+1}$ (resp. $F_{i,i+1}; K_i K_{i+1}^{-1}$) for $\alpha_i = \epsilon_i - \epsilon_{i+1}$.

Remark: (1) For each pair of indices $(i, j) \in \mathcal{I}$, the notation E_{ij}, F_{ij} are defined by

$$\begin{aligned} E_{ij} &= E_{ic} E_{cj} - q_c^{-1} E_{cj} E_{ic}, \\ F_{ij} &= -q_c F_{ic} F_{cj} + F_{cj} F_{ic}, \end{aligned} \quad i < c < j.$$

The relation (R2) then implies that, for $s \in [1, m+n]$, $(i, j) \in \mathcal{I}$,

$$\begin{aligned} K_s E_{ij} K_s^{-1} &= q_s^{\delta_{si} - \delta_{sj}} E_{ij} \\ K_s F_{ij} K_s^{-1} &= q_s^{-(\delta_{si} - \delta_{sj})} F_{ij}. \end{aligned}$$

(2) The parity of the elements $E_{ij}, F_{ij}, K_s^{\pm 1}$ is defined by $\bar{E}_{ij} = \bar{F}_{ij} = \bar{e}_{ij} \in \mathbb{Z}_2$, $\bar{K}_s^{\pm 1} = \bar{0}$.

(3) The bracket product in $U_q(\mathfrak{g})$ is defined by

$$[x, y] = xy - (-1)^{\bar{x}\bar{y}} yx, x, y \in h(U_q(\mathfrak{g})).$$

A bijective (even) \mathbb{F} -linear map f from an \mathbb{F} -superalgebra \mathfrak{A} into itself is called an anti-automorphism (resp. \mathbb{Z}_2 -graded anti-automorphism) if

$$f(xy) = f(y)f(x) \text{ (resp. } f(xy) = (-1)^{\bar{x}\bar{y}} f(y)f(x))$$

for any $x, y \in h(\mathfrak{A})$.

It is easy to show that

Lemma 3.1. [10, 12] *There is an anti-automorphism Ω and a \mathbb{Z}_2 -graded anti-automorphism Ψ of $U_q(\mathfrak{g})$ such that*

$$\Omega(E_{\alpha_i}) = F_{\alpha_i}, \Omega(F_{\alpha_i}) = E_{\alpha_i}, \Omega(K_j) = K_j^{-1}, \Omega(q) = q^{-1}$$

$$\Psi(E_{\alpha_i}) = E_{\alpha_i}, \Psi(F_{\alpha_i}) = F_{\alpha_i}, \Psi(K_j) = K_j, \Psi(q) = q^{-1},$$

for all $i \in [1, m+n], j \in [1, m+n]$.

From the lemma it is easily seen that

$$\Omega(E_{ij}) = F_{ij}, \Psi(E_{ij}) = q^z E_{ij}, \Psi(F_{ij}) = q^z F_{ij}, z \in \mathbb{Z}$$

for any $(i, j) \in \mathcal{I}$.

We abbreviate $U_q(\mathfrak{g})$ to U_q in the following.

4 Some formulas in U_q

In this section we present some formulas in U_q , most of which are given in [12]. To keep the paper self-contained, each formula will be proved unless an explicit proof can be found elsewhere.

For $i \in [1, m+n] \setminus m$, the automorphism T_{α_i} of U_q is defined by (see [12, Appendix A] and also [8, 1.3])

$$T_{\alpha_i}(E_{\alpha_j}) = \begin{cases} -F_{\alpha_i} K_{\alpha_i}, & \text{if } i = j \\ E_{\alpha_j}, & \text{if } |i - j| > 1 \\ -E_{\alpha_i} E_{\alpha_j} + q_i^{-1} E_{\alpha_j} E_{\alpha_i}, & \text{if } |i - j| = 1. \end{cases}$$

$$T_{\alpha_i} F_{\alpha_j} = \begin{cases} -K_{\alpha_i}^{-1} E_{\alpha_i}, & \text{if } i = j \\ F_{\alpha_j}, & \text{if } |i - j| > 1 \\ -F_{\alpha_j} F_{\alpha_i} + q_i F_{\alpha_i} F_{\alpha_j}, & \text{if } |i - j| = 1. \end{cases}$$

$$T_{\alpha_i} K_j = \begin{cases} K_{i+1}, & \text{if } j = i \\ K_i, & \text{if } j = i + 1 \\ K_j, & \text{if } j \neq i, i + 1. \end{cases}$$

T_{α_i} is an even automorphism for U_q , that is,

$$T_{\alpha_i}(uv) = T_{\alpha_i}(u)T_{\alpha_i}(v), \quad \text{for all } u, v \in h(U_q).$$

By a straightforward computation ([12, A3]), one obtains for each $i \in [1, m+n] \setminus m$ the inverse map $T_{\alpha_i}^{-1}$:

$$T_{\alpha_i}^{-1} E_{\alpha_j} = \begin{cases} -K_{\alpha_i}^{-1} F_{\alpha_i}, & \text{if } i = j \\ E_{\alpha_j}, & \text{if } |i - j| > 1 \\ -E_{\alpha_j} E_{\alpha_i} + q_i^{-1} E_{\alpha_i} E_{\alpha_j}, & \text{if } |i - j| = 1. \end{cases}$$

$$T_{\alpha_i}^{-1}F_{\alpha_j} = \begin{cases} -E_{\alpha_i}K_{\alpha_i}, & \text{if } i = j \\ F_{\alpha_j}, & \text{if } |i - j| > 1 \\ -F_{\alpha_i}F_{\alpha_j} + q_iF_{\alpha_j}F_{\alpha_i}, & \text{if } |i - j| = 1. \end{cases}$$

$$T_{\alpha_i}^{-1}K_j = \begin{cases} K_{i+1}, & \text{if } j = i \\ K_i, & \text{if } j = i + 1 \\ K_j, & \text{if } j \neq i, i + 1. \end{cases}$$

It follows from the definition that

$$(b1) \quad E_{ij} = (-1)^{j-i-1}T_{\alpha_i}T_{\alpha_{i+1}} \cdots T_{\alpha_{j-1}}E_{j-1,j} \\ = (-1)^{j-i-1}T_{\alpha_{j-1}}^{-1}T_{\alpha_{j-2}}^{-1} \cdots T_{\alpha_{i+1}}^{-1}E_{i,i+1},$$

$$(b2) \quad F_{i,j} = (-1)^{j-i-1}T_{\alpha_i}T_{\alpha_{i+1}} \cdots T_{\alpha_{j-1}}F_{j-1,j} \\ = (-1)^{j-i-1}T_{\alpha_{j-1}}^{-1}T_{\alpha_{j-2}}^{-1} \cdots T_{\alpha_{i+1}}^{-1}F_{i,i+1}.$$

By the defining relation (3), (4) and the formulas above we get

$$(1) \quad E_{ij}^2 = F_{ij}^2 = 0, \quad (i, j) \in \mathcal{I}_1,$$

$$(2)([12]) \quad [E_{ij}, F_{ij}] = \frac{K_iK_j^{-1} - K_i^{-1}K_j}{q_i - q_i^{-1}}, (i, j) \in \mathcal{I}.$$

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace over a field \mathbb{F} . A \mathbb{F} -linear mapping $f : V \rightarrow V$ is called \mathbb{Z}_2 -graded with parity $\bar{f} = \bar{i} \in \mathbb{Z}_2$ if $f(V_{\bar{k}}) \subseteq V_{\bar{k}+\bar{i}}$ for any $\bar{k} \in \mathbb{Z}_2$. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an associative \mathbb{F} -superalgebra. A \mathbb{Z}_2 -graded \mathbb{F} -linear mapping δ from A into itself is called a derivation if

$$\delta(xy) = \delta(x)y + (-1)^{\bar{\delta}\bar{x}}x\delta(y) \quad \text{for any } x, y \in h(A).$$

Denote by $\text{Der}_{\mathbb{F}}A$ the set of all derivations on A . For any $x, y \in h(A)$, we define $[x, y] = xy - (-1)^{\bar{x}\bar{y}}yx$. Clearly we have

$$[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x].$$

For each $x \in h(A)$, it is easy to see that $[x, -], [-, x] \in \text{Der}_{\mathbb{F}}A$.

Lemma 4.1. ([12]) *The following identities hold in U_q .*

$$(1) \quad F_{sj}F_{si} = (-1)^{\bar{F}_{si}}q_sF_{si}F_{sj}, s < i < j,$$

$$(2) \quad F_{is}F_{js} = (-1)^{\bar{F}_{js}}q_s^{-1}F_{js}F_{is}, i < j < s.$$

For $c < i < j$,

$$(3) \quad [F_{cj}, E_{ci}] = F_{ij}K_cK_i^{-1}q_i, (4) \quad [F_{ci}, E_{cj}] = E_{ij}K_c^{-1}K_i,$$

$$(5) \quad [E_{ij}, F_{cj}] = F_{ci}K_i^{-1}K_j, (6) \quad [E_{cj}, F_{ij}] = E_{ci}K_iK_j^{-1}q_i^{-1}.$$

$$(7) \quad [F_{st}, F_{ij}] = -(q_j - q_j^{-1})F_{sj}F_{it}, \quad i < s < j < t.$$

Proof. (1) and (2) follow from a short computation using the formulas provided by Remark (1) in Sec. 3.1.

(3) By Remark (1) in Sec. 3.1, we have

$$[F_{cj}, E_{ci}] = [F_{ij}F_{ci} - q_i F_{ci}F_{ij}, E_{ci}].$$

Since $[-, E_{ci}]$ is a derivation on U_q and $[F_{ij}, E_{ci}] = 0$, we have

$$[F_{cj}, E_{ci}] = F_{ij}[F_{ci}, E_{ci}] - q_i(-1)^{\bar{E}_{ci}\bar{F}_{ij}}[F_{ci}, E_{ci}]F_{ij}.$$

Let us note that at least one of the $\bar{E}_{ci}, \bar{F}_{ij}$ is $\bar{0} \in \mathbb{Z}_2$. Then Using the formula (2) we have that

$$\begin{aligned} [F_{cj}, E_{ci}] &= -(-1)^{\bar{E}_{ci}\bar{F}_{ci}}[F_{ij} \frac{K_c K_i^{-1} - K_c^{-1} K_i}{q_c - q_c^{-1}} - q_i \frac{K_c K_i^{-1} - K_c^{-1} K_i}{q_c - q_c^{-1}} F_{ij}] \\ &= F_{ij} K_c K_i^{-1} q_i (-1)^{\bar{E}_{ci}\bar{F}_{ci}} \frac{q_i - q_i^{-1}}{q_c - q_c^{-1}} \\ &= F_{ij} K_c K_i^{-1} q_i. \end{aligned}$$

It is easy to see that $\Omega([x, y]) = [\Omega(y), \Omega(x)]$ for any $x, y \in h(U_q)$, applying which to (3) we obtain (4).

(5),(6) can be proved similarly.

(7) follows from an application of Ω to [10, Lemma 4.2(6)]. □

Lemma 4.2. [10]

- (1) $[F_{ij}, F_{st}] = 0, \quad i < s < t < j,$
- (2) $[E_{ij}, F_{st}] = 0, \quad i < s < t < j,$
- (3) $[F_{ij}, E_{st}] = 0, \quad i < s < t < j.$

Lemma 4.3. For $i < s < j < t$, we have

- (a) $[E_{ij}, F_{st}] = (q_j^{-1} - q_j)(K_s K_j^{-1}) F_{jt} E_{is},$
- (b) $[E_{st}, F_{ij}] = (q_j - q_j^{-1}) F_{is} E_{jt} K_s^{-1} K_j.$

Proof. It suffices to prove (a), (b) follows from the application of Ω to (a). Since $[E_{ij}, -]$ is a derivation of U_q , we have

$$\begin{aligned} [E_{ij}, F_{st}] &= [E_{ij}, F_{jt} F_{sj} - q_j F_{sj} F_{jt}] \\ &= F_{jt} [E_{ij}, F_{sj}] - q_j [E_{ij}, F_{sj}] F_{jt} \\ (\text{Using Lemma 4.1(6)}) &= F_{jt} E_{is} K_s K_j^{-1} q_s^{-1} - q_j E_{is} K_s K_j^{-1} q_s^{-1} F_{jt} \\ &= (q_j^{-1} - q_j)(K_s K_j^{-1}) F_{jt} E_{is}. \end{aligned}$$

□

5 The simplicity of Kac modules

There is an order \prec defined on the set of elements $E_{ij}, (i, j) \in \mathcal{I}([10])$:

$$E_{ij} \prec E_{st} \quad \text{if} \quad (i, j) \in \mathcal{I}_0 \quad \text{and} \quad (s, t) \in \mathcal{I}_1$$

or

$$(i, j), (s, t) \in \mathcal{I}_\theta, \theta = 0, 1, i < s \quad \text{or} \quad i = s \quad \text{and} \quad j < t,$$

$$F_{ij} \prec F_{st} \quad \text{if and only if} \quad E_{ij} \succ E_{st}.$$

For each $\delta \in \{0, 1\}^{\mathcal{I}_1}$, let E_1^δ denote the product $\prod_{(i,j) \in \mathcal{I}_1} E_{ij}^{\delta_{ij}}$ in the order given above. Let $F_1^\delta = \Omega(E_1^\delta)$.

Set

$$\mathcal{N}_1 = \langle E_1^\delta | \delta \in \{0, 1\}^{\mathcal{I}_1} \rangle, \mathcal{N}_{-1} = \langle F_1^\delta | \delta \in \{0, 1\}^{\mathcal{I}_1} \rangle,$$

$$\mathcal{N}_{-1}^+ = \langle F_1^\delta | \sum \delta_{ij} > 0 \rangle, \mathcal{N}_1^+ = \langle E_1^\delta | \sum \delta_{ij} > 0 \rangle.$$

By [10], these are subalgebras of U_q , and

$$\begin{aligned} U_q &= \mathcal{N}_{-1} U_q(\mathfrak{g}_{\bar{0}}) \mathcal{N}_1 \\ &\cong \mathcal{N}_{-1} \otimes U_q(\mathfrak{g}_{\bar{0}}) \otimes \mathcal{N}_1. \end{aligned}$$

By applying the \mathbb{Z}_2 -graded anti-automorphism Ψ , we get

$$\begin{aligned} U_q &= \mathcal{N}_1 U_q(\mathfrak{g}_{\bar{0}}) \mathcal{N}_{-1} \\ &\cong \mathcal{N}_1 \otimes U_q(\mathfrak{g}_{\bar{0}}) \otimes \mathcal{N}_{-1}. \end{aligned}$$

The subalgebra $U_q(\mathfrak{g}_{\bar{0}}) \mathcal{N}_1$ (resp. $\mathcal{N}_{-1} U_q(\mathfrak{g}_{\bar{0}})$) has a nilpotent ideal $U_q(\mathfrak{g}_{\bar{0}}) \mathcal{N}_1^+$ (resp. $\mathcal{N}_{-1}^+ U_q(\mathfrak{g}_{\bar{0}})$), by which each simple $U_q(\mathfrak{g}_{\bar{0}})$ -module is annihilated. Therefore, each simple $U_q(\mathfrak{g}_{\bar{0}}) \mathcal{N}_1$ -module can be identified with a simple $U_q(\mathfrak{g}_{\bar{0}})$ -module (cf. [10]).

Let U^0 be the subalgebra of U_q generated by the elements $K_i^{\pm 1}, i \in [1, m+n]$. Then by the PBW theorem ([10]), U^0 is a polynomial algebra in variables $K_i^{\pm 1}, i \in [1, m+n]$. Let $K^\mu = \prod_{i=1}^{m+n} K_i^{\mu_i}$ for $\mu = \sum_{i=1}^{m+n} \mu_i \epsilon_i \in \Lambda$. Denote

$$X(U^0) =: \text{Hom}_{\mathbb{C}(q)\text{-alg}}(U^0, \mathbb{C}(q)).$$

Each $\lambda \in X(U^0)$ is completely determined by $\lambda(K_i) \in \mathbb{C}(q)^*, i \in [1, m+n]$. Then $X(U^0)$ is an additive group with the addition defined by

$$(\lambda_1 + \lambda_2)(K^\mu) = \lambda_1(K^\mu) + \lambda_2(K^\mu), \mu \in \Lambda.$$

Each $\lambda \in X(U^0)$ is called a weight for U_q . Note that Λ can be canonically imbedded in $X(U^0)$ by letting

$$\mu(K_i) = q_i^{\mu_i}, i \in [1, m+n], \mu = \sum_{i=1}^{m+n} \mu_i \epsilon_i.$$

A weight in Λ is called *integral*. Clearly we have $\lambda(K^\mu) = q^{(\lambda, \mu)}$ for $\lambda, \mu \in \Lambda$.

Let M be a $U_q(\mathfrak{g}_0)$ -module and let $\lambda \in X(U^0)$, let

$$M_\lambda = \{x \in M | ux = \lambda(u)x, u \in U^0\}.$$

A nonzero vector $v \in M_\lambda$ is called a maximal vector of weight λ if $E_{ij}v^+ = 0$ for all $(i, j) \in \mathcal{I}_0$. If M is finite dimensional, then $M = \sum M_\lambda$ ([6, Prop. 5.1]). If M is a finite dimensional simple $U_q(\mathfrak{g}_0)$ -module, then there is a maximal vector, unique up to scalar multiple, which generates M . In this case we denote M by $M(\lambda)$. Regard $M(\lambda)$ as a $U_q(\mathfrak{g}_0)\mathcal{N}_1$ -module annihilated by $U_q(\mathfrak{g}_0)\mathcal{N}_1^+$. Define the Kac module

$$K(\lambda) = U_q \otimes_{U_q(\mathfrak{g}_0)\mathcal{N}_1} M(\lambda).$$

Then we have $K(\lambda) = \mathcal{N}_{-1} \otimes_{\mathbb{F}} M(\lambda)$ as \mathcal{N}_{-1} -modules.

To study the simplicity of $K(\lambda)$, we define a new order on \mathcal{I}_1 by

$$(i, j) \prec (s, t) \quad \text{if } j > t \text{ or } j = t \text{ but } i < s.$$

We denote $(i, j) \preceq (s, t)$ if $(i, j) \prec (s, t)$ or $(i, j) = (s, t)$. We define $F_{ij} \prec F_{st}$ if $(i, j) \prec (s, t)$.

For each subset $I \subseteq \mathcal{I}_1$, denote by F_I the product $\prod_{(i, j) \in I} F_{ij}$ in the new order. In particular, we let $F_\emptyset = 1$. For each $I \subseteq \mathcal{I}_1$, set $E_I = \Omega(F_I)$.

For each $(i, j) \in \mathcal{I}_1$, denote by $> (i, j)$ (resp. $\geq (i, j)$; $< (i, j)$; $\leq (i, j)$) the subset

$$\begin{aligned} & \{(s, t) \in \mathcal{I}_1 | (s, t) \succ (i, j)\} \\ & \text{(resp. } \{(s, t) \in \mathcal{I}_1 | (s, t) \succeq (i, j)\}; \\ & \{(s, t) \in \mathcal{I}_1 | (s, t) \prec (i, j)\}; \\ & \{(s, t) \in \mathcal{I}_1 | (s, t) \preceq (i, j)\}). \end{aligned}$$

For $(i, j), (s, t) \in \mathcal{I}_1$ with $(i, j) \prec (s, t)$, set

$$((i, j), (s, t)) = \{(i', j') \in \mathcal{I}_1 | (i, j) \prec (i', j') \prec (s, t)\}.$$

Then we have

$$F_{>(m, m+1)} = F_{<(1, m+n)} = 1 \quad \text{and} \quad F_{\mathcal{I}_1} = F_{<(i, j)} F_{\geq(i, j)} = F_{\leq(i, j)} F_{>(i, j)}$$

for any $(i, j) \in \mathcal{I}_1$.

Lemma 5.1. (a) \mathcal{N}_{-1} (resp. \mathcal{N}_{-1}^+) has a basis F_I , $I \subseteq \mathcal{I}_1$ (resp. $\emptyset \neq I \subseteq \mathcal{I}_1$).

(b) \mathcal{N}_1 (resp. \mathcal{N}_1^+) has a basis E_I , $I \subseteq \mathcal{I}_1$ (resp. $\emptyset \neq I \subseteq \mathcal{I}_1$).

Proof. Since $\mathcal{N}_1 = \Omega(\mathcal{N}_{-1})$, (b) follows from the application of Ω to (a).

(a). Clearly the number of the above elements is equal to $\dim \mathcal{N}_{-1}$. We only need to show that the elements F_I span \mathcal{N}_{-1} .

First we claim that any product $F_{ij}F_{st}$, $(i, j), (s, t) \in \mathcal{I}_1$ can be written as an $\mathbb{Z}[q, q^{-1}]$ -linear combination of products in the new order. The case where $j = t$ and $i > s$ follows from Lemma 4.1(2). The cases where $j < t$ and $i \geq s$ follow from Lemma 4.1(1) and Lemma 4.2(1). The only case left is $i < s \leq m < j < t$, in which we have by Lemma 4.1(7) that

$$\begin{aligned} F_{ij}F_{st} &= -F_{st}F_{ij} - (q_j - q_j^{-1})F_{sj}F_{it} \\ (\text{Using Lemma 4.2(1)}) &= -F_{st}F_{ij} + (q_j - q_j^{-1})F_{it}F_{sj}. \end{aligned}$$

Thus, the claim follows.

Since \mathcal{I}_1 is a finite set, by induction on the cardinality $|I|$ of I we obtain that each product $\prod_{(i,j) \in I \subseteq \mathcal{I}_1} F_{ij}$ in any order can be written as a $\mathbb{Z}[q, q^{-1}]$ -linear combination of elements $F_{I'}, I' \subseteq \mathcal{I}_1$. \square

By the lemma, each element in $K(\lambda)$ is in the form $\sum_{I \subseteq \mathcal{I}_1} F_I \otimes v_I, v_I \in M(\lambda)$.

Lemma 5.2. *Let $(i, k) \in \mathcal{I}_1$. Then $F_{st}F_{\geq(i,k)} = 0$ for any $F_{st} \succeq F_{ik}$ or, equivalently $F_{st}F_{>(i,k)} = 0$ for any $F_{st} \succ F_{ik}$.*

Proof. Denote the set $\geq (i, k)$ by I . We proceed with induction on $|I|$. The case $|I| = 1$ is trivial. Assume the lemma for $|I| < d$ and consider the case $|I| = d > 1$.

Note that $F_I = F_{i,k}F_{>(i,k)}$. By Lemma 4.1(2) and the formula (1) in the preceding section, we have $F_{st}F_I = 0$ for any $(s, t) \in \mathcal{I}_1$ with $t = k$.

Suppose $t < k$. If $s \geq i$, by Lemma 4.1(2) and the formulas (1) in Sec. 4 we have

$$F_{st}F_I = \pm q^z F_{ik}F_{st}F_{>(i,k)} = 0, z \in \mathbb{Z},$$

where the last equality is given by the induction hypothesis.

If $s < i$, then we must have $s < i \leq m < t < k$. Note that $F_{it} \succ F_{ik}$ and $F_{st} \succ F_{ik}$. Then using Lemma 4.1(7) and the induction hypothesis we obtain

$$F_{st}F_I = -F_{ik}F_{st}F_{>(i,k)} + (q_t - q_t^{-1})F_{sk}F_{it}F_{>(i,k)} = 0.$$

\square

By a similar proof we can show that

Lemma 5.3. *Let $(i, k) \in \mathcal{I}_1$. Then $F_{\leq(i,k)}F_{st} = 0$ for any $F_{st} \preceq F_{ik}$ or, equivalently, $F_{<(i,k)}F_{st} = 0$ for any $F_{st} \prec F_{ik}$.*

Proposition 5.4. *For every $(i, j) \in \mathcal{I}_1$, there are $z_1, z_2 \in \mathbb{Z}$ such that*

$$\begin{aligned} (1) \quad & F_{ij}F_{<(i,j)} = \pm q^{z_1} F_{\leq(i,j)} \\ (2) \quad & F_{>(i,j)}F_{ij} = \pm q^{z_2} F_{\geq(i,j)}. \end{aligned}$$

Proof. (1) Let $(k, s) \in \mathcal{I}_1$. By Lemma 4.1, 4.2 we have,

$$F_{i,j}F_{k,s} = \begin{cases} -F_{ks}F_{i,j}, & \text{if } k < i \text{ and } s > j \\ q_k F_{ks}F_{i,j}, & \text{if } i = k \text{ and } s > j \\ -F_{ks}F_{i,j} + (q_j - q_j^{-1})F_{i,s}F_{kj}, & \text{if } i < k < j < s. \end{cases}$$

Let $(k_1, s_1), \dots, (k_p, s_p)$ be all the pairs in the set $\leq (i, j)$ such that $i < k_t < j < s_t, t = 1, \dots, p$, so that $F_{i,s_t} \prec F_{k_t,s_t}$. Then there are integers z'_1, \dots, z'_p such that

$$\begin{aligned} F_{ij}F_{<(i,j)} &= F_{ij}F_{<(k_1,s_1)}F_{k_1,s_1}F_{((k_1,s_1),(i,j))} \\ &= \pm q^{z'_1}F_{<(k_1,s_1)}(F_{ij}F_{k_1,s_1})F_{((k_1,s_1),(i,j))} \\ &= \pm q^{z'_1}F_{<(k_1,s_1)}(-F_{k_1,s_1}F_{ij} + (q_j - q_j^{-1})F_{i,s_1}F_{k_1,j})F_{((k_1,s_1),(i,j))} \\ (\text{Using Lemma 5.3}) &= \pm q^{z'_1}F_{\leq(k_1,s_1)}F_{ij}F_{((k_1,s_1),(i,j))} \\ &= \dots \\ &= \pm q^{z'_p}F_{<(i,j)}F_{ij} \\ &= \pm q^{z'_p}F_{\leq(i,j)}. \end{aligned}$$

(2) can be verified similarly. \square

As an immediate consequence, we have

Corollary 5.5. *Let $(i, j), (s, t) \in \mathcal{I}_1$ with $(s, t) \preceq (i, j)$. Then $F_{st}F_{\leq(i,j)} = 0$.*

Lemma 5.6. *Each nonzero submodule of $K(\lambda)$ contains $F_{\mathcal{I}_1} \otimes v$ for some $0 \neq v \in M(\lambda)$.*

Proof. Let I, I' be two nonempty subsets of \mathcal{I}_1 . We define $I < I'$ if, with respect to the order in \mathcal{I}_1 , the first pair $(s, t) \notin I \cap I'$ is in I' . Then we have by Prop. 5.4 that $F_{st}F_I = \pm q^z F_{I \cap (s,t)}$ for some $z \in \mathbb{Z}$ and $F_{st}F_{I'} = 0$.

Let $N = N_{\bar{0}} \oplus N_{\bar{1}}$ be a nonzero submodule of $K(\lambda)$. Take a nonzero element $x = \sum_{I \subseteq \mathcal{I}_1} F_I \otimes v_I \in N$, $v_I \neq 0$ for all I . Let \bar{I} be the minimal subset appeared in the expression.

We proceed with induction on the order of \bar{I} . If $\bar{I} = \mathcal{I}_1$, that is, $x = F_{\mathcal{I}_1} \otimes v$, the lemma follows. Suppose $\bar{I} \neq \mathcal{I}_1$. Let $(s, t) \in \mathcal{I}_1$ be the first pair such that $(s, t) \notin \bar{I}$. Then by definition we have $(i, j) \in I$ for all $(i, j) \prec (s, t)$ and all I appeared above. Applying F_{st} to x and using Prop. 5.4, we have $F_{st}x \neq 0$, and the minimal I appeared in $F_{st}x$, denoted \bar{I}' , satisfies $\bar{I}' > \bar{I}$. Then the induction hypothesis yields the lemma. \square

Lemma 5.7. *For any $(i, j) \in \mathcal{I}_0$, there is $z \in \mathbb{Z}$ such that $F_{ij}F_{\mathcal{I}_1} = q^z F_{\mathcal{I}_1}F_{ij}$.*

Proof. Recall that $F_{ij} = -q_c F_{ic}F_{cj} + F_{cj}F_{ic}$, $i < c < j$. Then it suffices to consider the case $j = i + 1$.

By Lemma 4.1(1), (2) and Lemma 4.2(1) we have

$$F_{i,i+1}F_{sk} = \begin{cases} q_k F_{sk} F_{k,k+1} + F_{s,k+1}, & \text{if } i = k \\ q_{i+1}^{-1} (F_{i+1,k} F_{i,i+1} - F_{ik}), & \text{if } s = i + 1 \\ q^z F_{sk} F_{i,i+1}, & \text{otherwise,} \end{cases}$$

for some $z \in \mathbb{Z}$. Since $(i, i+1) \in \mathcal{I}_0$, we have that $F_{i,i+1}$ commutes, up to multiple of $q^z, z \in \mathbb{Z}$, with all $F_{sk}, (s, k) \in \mathcal{I}_1$, but the case $s = i + 1$ if $i < m$ and the case $i = k$ if $i > m$.

Assume $i < m$. Then we have

$$\begin{aligned} F_{i,i+1}F_{\mathcal{I}_1} &= F_{i,i+1}F_{<(i+1,m+n)}F_{i+1,m+n}F_{>(i+1,m+n)} \\ &= q^{z_1}F_{<(i+1,m+n)}(F_{i,i+1}F_{i+1,m+n})F_{>(i+1,m+n)} \\ &= q^{z_1-1}F_{<(i+1,m+n)}(F_{i+1,m+n}F_{i,i+1} - F_{i,m+n})F_{>(i+1,m+n)} \\ (\text{Using } F_{<(i+1,m+n)}F_{i,m+n} &= 0) = q^{z_1-1}F_{\leq(i+1,m+n)}F_{i,i+1}F_{>(i+1,m+n)} \\ &= q^{z_2}F_{<(i+1,m+n-1)}(F_{i,i+1}F_{i+1,m+n-1})F_{>(i+1,m+n-1)} \\ &= \dots \\ &= q^z F_{\mathcal{I}_1} F_{i,i+1}. \end{aligned}$$

Similarly one verifies that $F_{i,i+1}F_{\mathcal{I}_1} = q^z F_{\mathcal{I}_1} F_{i,i+1}$ for some $z \in \mathbb{Z}$, if $i > m$. This completes the proof. \square

Lemma 5.8. *For any $(i, j) \in \mathcal{I}_0$, we have $E_{ij}F_{\mathcal{I}_1} = F_{\mathcal{I}_1}E_{ij}$.*

Proof. By the formula $E_{ij} = E_{ic}E_{cj} - q_c^{-1}E_{cj}E_{ic}$, it suffices to assume $j = i + 1$. Recall the (even) derivation $[E_{i,i+1}, -]$ of U_q .

Using the definition of U_q and Lemma 4.1(1), (2) we have, for any $(s, k) \in \mathcal{I}_1$,

$$[E_{i,i+1}, F_{sk}] = \begin{cases} -F_{i+1,k}K_iK_{i+1}^{-1}q_{i+1}, & \text{if } i = s \\ F_{si}K_i^{-1}K_{i+1}, & \text{if } i + 1 = k \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} &[E_{i,i+1}, F_{\mathcal{I}_1}] \\ &= \sum_{(s,k) \in \mathcal{I}_1} F_{<(s,k)}[E_{i,i+1}, F_{sk}]F_{>(s,k)} \\ &= \begin{cases} \sum_{s=i} F_{<(s,k)}(-F_{i+1,k}K_iK_{i+1}^{-1}q_{i+1})F_{>(s,k)}, & \text{if } i < m \\ \sum_{k=i+1} F_{<(s,k)}(F_{si}K_i^{-1}K_{i+1})F_{>(s,k)}, & \text{if } i > m \end{cases} \\ &= 0. \end{aligned}$$

where the last equality is given by the fact that $F_{i+1,k} \succ F_{s,k}$ if $s = i$ and $F_{si} \succ F_{s,k}$ if $k = i + 1$. Then the lemma follows. \square

Let $E_{\mathcal{I}_1} = \Omega(F_{\mathcal{I}_1})$. Using the triangular decomposition $U_q = U_q^- \otimes U^0 \otimes U_q^+$ we have

$$E_{\mathcal{I}_1} F_{\mathcal{I}_1} = f(K) + \sum u_i^- u_i^0 u_i^+, u_i^\pm \in U_q^\pm, f(K), u_i^0 \in U^0.$$

Note that U_q is a U^0 -module under the conjugation:

$$K_i \cdot u = K_i u K_i^{-1}, 1 \leq i \leq m+n.$$

Since the U^0 -weight of $E_{\mathcal{I}_1} F_{\mathcal{I}_1}$ is zero, we get $u_i^+ = 0$ if and only if $u_i^- = 0$.

Let v_λ be a maximal vector in $M(\lambda) \subseteq K(\lambda)$. Then we get

$$E_{\mathcal{I}_1} F_{\mathcal{I}_1} v_\lambda = f(K) v_\lambda = f(K)(\lambda) v_\lambda, f(K)(\lambda) \in C(q).$$

As $\lambda \in X(U^0)$ varies, one obtains a function $f(K)(\lambda)$. We denote it by $f_{m,n}(\lambda)$.

Proposition 5.9. *The U_q -module $K(\lambda)$ is simple if and only if $f_{m,n}(\lambda) \neq 0$.*

Proof. Assume $f_{m,n}(\lambda) \neq 0$. Let $N = N_{\bar{0}} \oplus N_{\bar{1}}$ be a nonzero submodule of $K(\lambda)$. By Lemma 5.6, we have $F_{\mathcal{I}_1} \otimes v \in N$ for some $0 \neq v \in M(\lambda)$. Since $K_i F_{\mathcal{I}_1} = q^{a_i} F_{\mathcal{I}_1} K_i$ for some $a_i \in \mathbb{Z}$, we may assume v is a weight vector. Since $M(\lambda)$ contains a unique (up to scalar multiple) maximal vector v_λ , there is a sequence of elements $E_{\alpha_{i_1}}, \dots, E_{\alpha_{i_s}} \in U_q(\mathfrak{g}_{\bar{0}})$ such that

$$E_{\alpha_{i_1}} \cdots E_{\alpha_{i_s}} v = v_\lambda.$$

Then Lemma 5.8 implies that $F_{\mathcal{I}_1} \otimes v_\lambda \in N$, and hence $E_{\mathcal{I}_1} F_{\mathcal{I}_1} \otimes v_\lambda = f_{m,n}(\lambda) \otimes v_\lambda \in N$. It follows that $v_\lambda \in N$ and hence $N = K(\lambda)$, so that $K(\lambda)$ is simple.

Suppose $K(\lambda)$ is simple. By Lemma 5.7, 5.8, the subspace $F_{\mathcal{I}_1} \otimes M(\lambda) \subseteq K(\lambda)$ is a $U_q(\mathfrak{g}_{\bar{0}})$ -submodule, and hence simple. Note that Coro.5.5 says that $\mathcal{N}_{-1}^+ F_{\mathcal{I}_1} \otimes M(\lambda) = 0$, so that $F_{\mathcal{I}_1} \otimes M(\lambda)$ is a simple $\mathcal{N}_{-1} U_q(\mathfrak{g}_{\bar{0}})$ -module annihilated by $\mathcal{N}_{-1}^+ U_q(\mathfrak{g}_{\bar{0}})$. Since $K(\lambda)$ is simple, we have

$$K(\lambda) = \mathcal{N}_1 U_q(\mathfrak{g}_{\bar{0}}) \mathcal{N}_{-1} F_{\mathcal{I}_1} \otimes M(\lambda) = \mathcal{N}_1 F_{\mathcal{I}_1} \otimes M_0(\lambda).$$

Since $\dim \mathcal{N}_{-1} = \dim \mathcal{N}_1$, we have that $K(\lambda)$ has a basis

$$E_I F_{\mathcal{I}_1} \otimes v_i, I \subseteq \mathcal{I}_1, i = 1, \dots, s,$$

with v_1, \dots, v_s a basis of $M(\lambda)$. We can choose $v_1 = v_\lambda$. Then we get

$$0 \neq F_{\mathcal{I}_1} F_{\mathcal{I}_1} v_\lambda = f_{m,n}(\lambda) v_\lambda,$$

so that $f_{m,n}(\lambda) \neq 0$. □

6 The polynomial $f_{m,n}(\lambda)$

This section is devoted to the determination of the polynomial $f_{m,n}(\lambda)$, for $\lambda \in X(U^0)$. Let us note that R. Zhang defined in [12] a polynomial using a different order of the product $\prod_{(i,j) \in \mathcal{I}_1} F_{ij}$.

Lemma 6.1. *For $1 \leq i \leq m$, we have $E_{i,m+n} F_{>(i,m+n)} v_\lambda = 0$.*

Proof. Using the formulas from Lemma 4.1, 4.3, we have, for any $(s, t) \succ (i, m+n)$,

$$[E_{i,m+n}, F_{st}] = \begin{cases} E_{t,m+n} K_s^{-1} K_t, & \text{if } s = i, t < m+n \\ E_{is} K_s K_{m+n}^{-1} q_s^{-1}, & \text{if } s > i, t = m+n \\ (q_t - q_t^{-1}) F_{si} E_{t,m+n} K_i^{-1} K_t, & \text{if } s < i < t < m+n \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} E_{i,m+n} F_{>(i,m+n)} v_\lambda &= [E_{(i,m+n)}, F_{>(i,m+n)}] v_\lambda \\ &= \sum_{F_{st} \succ F_{i,m+n}} (-1)^{\alpha_{st}} F_{((i,m+n),(s,t))} [E_{i,m+n}, F_{st}] F_{>(s,t)} v_\lambda \\ &= \sum_{s > i, t = m+n} (-1)^{\alpha_{st}} F_{((i,m+n),(s,t))} (E_{is} K_s K_{m+n}^{-1} q_s^{-1}) F_{>(s,m+n)} v_\lambda \\ &\quad + \sum_{s=i, t < m+n} (-1)^{\alpha_{st}} F_{((i,m+n),(s,t))} (E_{t,m+n} K_s^{-1} K_t) F_{>(s,t)} v_\lambda \\ &\quad + \sum_{s < i < t < m+n} (-1)^{\alpha_{st}} F_{((i,m+n),(s,t))} ((q_t - q_t^{-1}) F_{si} E_{t,m+n} K_i^{-1} K_t) F_{>(s,t)} v_\lambda, \end{aligned}$$

where $\alpha_{st} \in \mathbb{Z}_2$. Note that the second and the third summation are equal to zero, since $E_{t,m+n}$ commutes with all $F_{ij} ((i, j) \in \mathcal{I}_1)$ with $F_{ij} \succ F_{st}$.

We claim that the first summation is also equal to zero. In fact, we have, in the case where $s > i, t = m+n$,

$$\begin{aligned} E_{is} F_{>(s,m+n)} v_\lambda &= [E_{is}, F_{>(s,m+n)}] v_\lambda \\ &= \sum_{j=m+n-1}^{m+1} \sum_{k=i}^{s-1} F_{((s,m+n),(k,j))} [E_{is}, F_{kj}] F_{>(k,j)} v_\lambda. \end{aligned}$$

For $k = i, m+1 \leq j \leq m+n-1$, we have by Lemma 4.1(3) that

$$[E_{is}, F_{kj}] F_{>(k,j)} v_\lambda = q_s F_{sj} (K_i K_s^{-1}) F_{>(k,j)} v_\lambda = 0,$$

where the last equality is given by the fact that $(s, j) \succ (k, j)$.

For $i < k \leq s-1$, we have by using Lemma 4.3(a) that

$$[E_{is}, F_{kj}] F_{>(k,j)} v_\lambda = (q_s^{-1} - q_s) (K_k K_s^{-1}) E_{ik} F_{sj} F_{>(k,j)} v_\lambda = 0,$$

where the last equality follows from the fact that $(s, j) \succ (k, j)$. Thus, the claim follows. \square

For $(i, j) \in \mathcal{I}$, let $K_{ij} = K_i K_j^{-1}$. Let us denote

$$[(\lambda + \rho)(K_{ij})] = \frac{(\lambda + \rho)(K_{ij}) - (\lambda + \rho)(K_{ij}^{-1})}{q - q^{-1}}.$$

Then we see that $[(\lambda + \rho)(K_{ij})] = [(\lambda + \rho, \epsilon_i - \epsilon_j)]$ if λ is integral.

Theorem 6.2. *Let $\lambda \in X(U^0)$. Then $f_{m,n}(\lambda) = \Pi_{(i,j) \in \mathcal{I}_1} [(\lambda + \rho)(K_{ij})]$. In particular, $f_{m,n}(\lambda) = \Pi_{(i,j) \in \mathcal{I}_1} [(\lambda + \rho, \epsilon_i - \epsilon_j)]$ if λ is integral.*

Proof. Using the formula (2) in Sec. 4, we have

$$\begin{aligned} E_{\mathcal{I}_1} F_{\mathcal{I}_1} v_\lambda &= E_{>(1,m+n)} (E_{1,m+n} F_{1,m+n}) F_{>(1,m+n)} v_\lambda \\ &= E_{>(1,m+n)} \left(\frac{K_{1,m+n} - K_{1,m+n}^{-1}}{q - q^{-1}} \right) F_{>(1,m+n)} v_\lambda \\ &\quad - E_{>(1,m+n)} F_{1,m+n} E_{1,m+n} F_{>(1,m+n)} v_\lambda \\ (\text{Using Lemma 6.1}) &= E_{>(1,m+n)} \frac{K_{1,m+n} - K_{1,m+n}^{-1}}{q - q^{-1}} F_{>(1,m+n)} v_\lambda \\ &= [(\lambda + \alpha_1)(K_{1,m+n})] E_{>(1,m+n)} F_{>(1,m+n)} v_\lambda, \end{aligned}$$

where $\lambda + \alpha_1$ is the weight of $F_{>(1,m+n)} v_\lambda$.

Next we compute $E_{>(1,m+n)} F_{>(1,m+n)} v_\lambda$ in a similar way. Continue the process, we get

$$\begin{aligned} E_{\mathcal{I}_1} F_{\mathcal{I}_1} v_\lambda &= [(\lambda + \alpha_1)(K_{1,m+n})] E_{>(1,m+n)} F_{>(1,m+n)} v_\lambda \\ &= [(\lambda + \alpha_1)(K_{1,m+n})] [(\lambda + \alpha_2)(K_{2,m+n})] E_{>(2,m+n)} F_{>(2,m+n)} v_\lambda \\ &= \dots \\ &= \Pi_{i=1}^m [(\lambda + \alpha_i)(K_{i,m+n})] E_{\geq(1,m+n-1)} F_{\geq(1,m+n-1)} v_\lambda, \end{aligned}$$

where $\lambda + \alpha_i$ is the weight of $F_{>(i,m+n)} v_\lambda$, $1 \leq i \leq m$. It is easily seen that

$$\lambda + \alpha_i = \lambda - 2\rho_1 + \sum_{k=1}^i (\epsilon_k - \epsilon_{m+n}).$$

By the proof of [11, Th.4], we have

$$(\alpha_i, \epsilon_i - \epsilon_{m+n}) = (\rho, \epsilon_i - \epsilon_{m+n}),$$

so that

$$(\lambda + \alpha_i)(K_{i,m+n}) = \lambda(K_{i,m+n}) q^{(\rho, \epsilon_i - \epsilon_{m+n})} = (\lambda + \rho)(K_{i,m+n})$$

for any $i \leq m$, which gives

$$f_{m,n}(\lambda) = \Pi_{k=1}^m [(\lambda + \rho)(K_{k,m+n})] E_{\geq(1,m+n-1)} F_{\geq(1,m+n-1)} v_\lambda.$$

We now prove the proposition by induction on n . The case $n = 1$ follows immediately from the equation above. Assume the proposition for $n-1$. To proceed, let us denote by $\rho_{m,n-1}$ the ρ for Lie superalgebra $gl(m, n-1)$. By the proof of [11, Th.4], we have

$$(\rho_{m,n-1}, \epsilon_i - \epsilon_j) = (\rho, \epsilon_i - \epsilon_j)$$

for $i < m < j \leq m+n-1$. Applying the induction hypothesis, we have

$$\begin{aligned} f_{m,n}(\lambda) &= \Pi_{k=1}^m [(\lambda + \rho)(K_{k,m+n})] f_{m,n-1}(\lambda) \\ &= \Pi_{k=1}^m [(\lambda + \rho)(K_{k,m+n})] \Pi_{i < m < j \leq m+n-1} [(\lambda + \rho_{m,n-1})(K_{ij})] \\ &= \Pi_{k=1}^m [(\lambda + \rho)(K_{k,m+n})] \Pi_{i < m < j \leq m+n-1} \frac{\lambda(K_{ij}) \rho_{m,n-1}(K_{ij}) - \lambda(K_{ij}^{-1}) \rho_{m,n-1}(K_{ij}^{-1})}{q - q^{-1}} \\ &= \Pi_{k=1}^m [(\lambda + \rho)(K_{k,m+n})] \Pi_{i < m < j \leq m+n-1} \frac{\lambda(K_{ij}) q^{(\rho_{m,n-1}, \epsilon_i - \epsilon_j)} - \lambda(K_{ij}^{-1}) q^{-(\rho_{m,n-1}, \epsilon_i - \epsilon_j)}}{q - q^{-1}} \\ &= \Pi_{(i,j) \in \mathcal{I}_1} [(\lambda + \rho)(K_{ij})]. \end{aligned}$$

□

7 Representations of U_q at roots of unity

7.1 Simple U_η -modules

Let l be an odd number ≥ 3 and let η be a primitive l th root of unity. For $1 \leq i \leq m$, let $\eta_i = \begin{cases} \eta, & \text{if } i \leq m \\ \eta^{-1}, & \text{if } i > m. \end{cases}$ Set

$$\mathcal{A}' = \{f(q)/g(q) \mid f(q), g(q) \in \mathcal{A}, g(\eta) \neq 0\}.$$

Let $U_{\mathcal{A}'}$ be the \mathcal{A}' -subalgebra of U_q generated by the elements

$$E_{i,i+1}, F_{i,i+1}, K_j^{\pm 1}, i \in [1, m+n), j \in [1, m+n].$$

For $\psi = (\psi_{ij}) \in \mathbb{N}^{\mathcal{I}_0}$, let E_0^ψ denote the product $\Pi_{(i,j) \in \mathcal{I}_0} E_{ij}^{\psi_{ij}}$ in the order given in Sec.5 and let $F_0^\psi = \Omega(E_0^\psi)$. Recall the notion $E_I, F_I, I \subseteq \mathcal{I}_1$. Then by Lemma 5.1 and the PBW theorem of U_q (see [10]) we have

Corollary 7.1. *The \mathcal{A}' -superalgebra $U_{\mathcal{A}'}$ has an \mathcal{A}' -basis*

$$F_I F_0^\psi K^\mu E_0^{\psi'} E_{I'}, I, I' \subseteq \mathcal{I}_1, \psi, \psi' \in \mathbb{N}^{\mathcal{I}_0}, \mu \in \Lambda.$$

Let $U_{\mathcal{A}'}(\mathfrak{g}_{\bar{0}})$ (resp. $\mathcal{N}_{1,\mathcal{A}'}; \mathcal{N}_{-1,\mathcal{A}'}$) be the \mathcal{A}' -subalgebra of $U_{\mathcal{A}'}$ generated by elements $E_{\alpha_i}, F_{\alpha_i}, K_{\alpha_j}^{\pm 1}, i \in [1, m+n) \setminus m, j \in [1, m+n]$ (resp. $E_{ij}, (i, j) \in \mathcal{I}_1; F_{ij}, (i, j) \in \mathcal{I}_1$). Then we have by Sec. 5 that

$$U_{\mathcal{A}'} = \mathcal{N}_{-1,\mathcal{A}'} U_{\mathcal{A}'}(\mathfrak{g}_{\bar{0}}) \mathcal{N}_{1,\mathcal{A}'}.$$

Moreover, we have from the above corollary that there is an \mathcal{A}' -module isomorphism;

$$U_{\mathcal{A}'} \cong \mathcal{N}_{-1,\mathcal{A}'} \otimes U_{\mathcal{A}'}(\mathfrak{g}_{\bar{0}}) \otimes \mathcal{N}_{1,\mathcal{A}'}.$$

Lemma 5.1 says that $\mathcal{N}_{-1,\mathcal{A}'}$ (resp. $\mathcal{N}_{1,\mathcal{A}'}$) has an \mathcal{A}' -basis F_I (resp. E_I), $I \subseteq \mathcal{I}_1$.

Let $\mathcal{N}_{1,\mathcal{A}'}^+$ (resp. $\mathcal{N}_{-1,\mathcal{A}'}^+$) be the \mathcal{A}' -submodule of \mathcal{N}_1 (resp. \mathcal{N}_{-1}) generated by elements E_I (resp. F_I), $I \neq \emptyset$. Then by [10] $\mathcal{N}_{1,\mathcal{A}'}^+$ (resp. $\mathcal{N}_{-1,\mathcal{A}'}^+$) is an \mathcal{A}' -subalgebra of $U_{\mathcal{A}'}$. Moreover, using the formulas from Sec. 4 it is easy to see that $U_{\mathcal{A}'}(\mathfrak{g}_{\bar{0}}) \mathcal{N}_{1,\mathcal{A}'}^+$ and $\mathcal{N}_{-1,\mathcal{A}'}^+ U_{\mathcal{A}'}(\mathfrak{g}_{\bar{0}})$ are \mathcal{A}' -subalgebras of $U_{\mathcal{A}'}$ having $U_{\mathcal{A}'}(\mathfrak{g}_{\bar{0}}) \mathcal{N}_{1,\mathcal{A}'}^+$ and $\mathcal{N}_{-1,\mathcal{A}'}^+ U_{\mathcal{A}'}(\mathfrak{g}_{\bar{0}})$ as nilpotent ideals respectively.

Let $B_{\mathcal{A}'}$ (resp. $B_{\mathcal{A}'}^-; U_{\mathcal{A}'}^0$) be the \mathcal{A}' -subalgebra of $U_{\mathcal{A}'}(\mathfrak{g}_{\bar{0}})$ generated by elements $E_{\alpha_i}, i \neq m$ (resp. $F_{\alpha_i}, i \neq m; K_i^{\pm 1}, 1 \leq i \leq m+n$). By [6, Th. 4.21], we have

$$U_{\mathcal{A}'}(\mathfrak{g}_{\bar{0}}) \cong B_{\mathcal{A}'} \otimes U_{\mathcal{A}'}^0 \otimes B_{\mathcal{A}'}^-.$$

Moreover, the \mathcal{A}' -algebra $B_{\mathcal{A}'}$ (resp. $B_{\mathcal{A}'}^-$) is the algebra generated by the elements E_{α_i} (resp. F_{α_i}), $i \neq m$ with relations (R5), (R6) (resp. (R5), (R7)). Set

$$\begin{aligned} U_{\eta} &= U_{\mathcal{A}'} \otimes_{\mathcal{A}'} \mathbb{C}, & U_{\eta}(\mathfrak{g}_{\bar{0}}) &= U_{\mathcal{A}'}(\mathfrak{g}_{\bar{0}}) \otimes_{\mathcal{A}'} \mathbb{C} \\ \mathcal{N}_{-1,\eta} &= \mathcal{N}_{-1,\mathcal{A}'} \otimes \mathbb{C}, & \mathcal{N}_{1,\eta} &= \mathcal{N}_{1,\mathcal{A}'} \otimes \mathbb{C} \\ \mathcal{N}_{1,\eta}^+ &= \mathcal{N}_{1,\mathcal{A}'}^+ \otimes \mathbb{C}, & \mathcal{N}_{-1,\eta}^+ &= \mathcal{N}_{-1,\mathcal{A}'}^+ \otimes \mathbb{C} \\ B_{\eta} &= B_{\mathcal{A}'} \otimes \mathbb{C}, & B_{\eta}^- &= B_{\mathcal{A}'}^- \otimes \mathbb{C}, \\ U_{\eta}^0 &= U_{\mathcal{A}'}^0 \otimes \mathbb{C}, \end{aligned}$$

where \mathbb{C} is viewed as an \mathcal{A}' -algebra with q acting as multiplication by η . Then $U_{\eta}(\mathfrak{g}_{\bar{0}}), \mathcal{N}_{\pm 1,\eta}, \mathcal{N}_{1,\eta}^+$ can be viewed as \mathbb{C} -subalgebras of U_{η} . We also have \mathbb{C} -algebra isomorphisms:

$$\begin{aligned} U_{\eta} &\cong \mathcal{N}_{-1,\eta} \otimes U_{\eta}(\mathfrak{g}_{\bar{0}}) \otimes \mathcal{N}_{1,\eta} \\ U_{\eta}(\mathfrak{g}_{\bar{0}}) &\cong B_{\eta}^- \otimes U_{\eta}^0 \otimes B_{\eta}. \end{aligned}$$

For $x \in U_{\mathcal{A}'}$, we denote $x \otimes 1 \in U_{\eta}$ also by x . Then B_{η} (resp. B_{η}^-) is the algebra generated by the elements E_{α_i} (resp. F_{α_i}), $i \neq m$ with relations (R5), (R6) (resp. (R5), (R7)) in which q is replaced by η .

Corollary 7.2. (PBW theorem) *The \mathbb{C} -superalgebra U_{η} has a basis*

$$F_I F_0^{\psi} K^{\mu} E_0^{\psi'} E_{I'}, I, I' \subseteq \mathcal{I}_1, \psi, \psi' \in \mathbb{N}^{\mathcal{I}_0}, \mu \in \Lambda.$$

The center of the \mathbb{C} -superalgebra U_{η} is defined by

$$Z(U_{\eta}) = \{x \in (U_{\eta})_{\bar{0}} | xu = ux \text{ for all } u \in U_{\eta}\}.$$

Let $(i, j) \in \mathcal{I}_0$, $s \in [1, m+n]$. Then it is easy to see that

$$x_{ij} =: E_{ij}^l, y_{ij} =: F_{ij}^l, z_s^{\pm 1} =: K_s^{\pm l}$$

are all contained in $Z(U_\eta)$. By the PBW theorem for U_η , the \mathbb{C} -subalgebra Z_0 generated by these elements is a polynomial algebra in variables $x_{ij}, y_{ij}, z_s^{\pm 1}$. Set

$$\Lambda_l =: \{k_1 \epsilon_1 + \cdots + k_{m+n} \epsilon_{m+n} \in \Lambda \mid 0 \leq k_i < l, i = 1, \dots, m+n\}.$$

Clearly we have

Lemma 7.3. U_η is a free Z_0 -module having a basis

$$F_I F_0^\psi K^\mu E_0^{\psi'} E_{I'}, I, I' \subseteq \mathcal{I}_1, \psi, \psi' \in [0, l)^{\mathcal{I}_0}, \mu \in \Lambda_l.$$

Let $M = M_0 \oplus M_{\bar{1}}$ be a simple U_η -module. For any $z \in Z_0$, we define a linear mapping

$$\phi_z : M \longrightarrow M, \phi_z(x) = zx, x \in M.$$

Clearly ϕ_z is an even U_η -module homomorphism. Since $\ker \phi_z$ is a \mathbb{Z}_2 -graded submodule of M , either $\ker \phi_z = M$ or $\ker \phi_z = 0$. In the former case, we have $\phi_z = 0$; in the latter case, the simplicity of M says that $\phi_z(M) = M$, so that ϕ_z is an (even) isomorphism.

Lemma 7.4. ([9, Lemma 2.1, Ch.5]) Let R be a commutative ring with unity and suppose that $I \subset R$ is an ideal of R . Let V be a finitely generated unitary R -module with annihilator $\text{ann}_R(V) = \{r \in R \mid rv = 0 \text{ for all } v \in V\}$. If $IV = V$, then $I + \text{ann}_R(V) = R$.

Proposition 7.5. Let $M = M_0 \oplus M_{\bar{1}}$ be a simple U_η -module. Then M is finite dimensional.

Proof. Let $V = V_0 \oplus V_{\bar{1}}$ be a simple U_η -module. Since U_η is a finitely generated Z_0 -module by Lemma 7.3, V is a finitely generated Z_0 -module. Given any ideal $I \subseteq Z_0$, IV is a U_η -submodule of V . Then either $IV = V$ or $IV = 0$. Since $1 \in Z_0$, $\text{ann}_{Z_0}(V) \neq Z_0$. Let $I \neq Z_0$ be any ideal containing $\text{ann}_{Z_0}(V)$. If $IV = V$, then by the above lemma we get $Z_0 = \text{ann}_{Z_0}(V) + I = I$, a contradiction. Therefore, we have $IV = 0$; that is $I = \text{ann}_{Z_0}(V)$, which implies that $\text{ann}_{Z_0}(V)$ is a maximal ideal of Z_0 . By Hilbert's nullstellensatz, $Z_0/\text{ann}_{Z_0}(V)$ is finite dimensional over \mathbb{C} . Since V is finite dimensional over $Z_0/\text{ann}_{Z_0}(V)$, V is finite dimensional over \mathbb{C} . \square

Lemma 7.6. For each simple U_η -module $V = V_0 \oplus V_{\bar{1}}$, there is a \mathbb{C} -algebra homomorphism $\chi : Z_0 \longrightarrow \mathbb{C}$ such that $(z - \chi(z))M = 0$ for any $z \in Z_0$.

Proof. Let $z \in Z_0$. Since \mathbb{C} is algebraically closed and V is finite dimensional, there is $\chi(z) \in \mathbb{C}$ and nonzero $v \in V$ such that $zv = \chi(z)v$. Then

$$V_\chi =: \{v \in V \mid zv = \chi(z)v\} \neq 0.$$

Since $z \in (U_\eta)_0$, V_χ is \mathbb{Z}_2 -graded. Clearly V_χ is a U_η -submodule of V . Thus, we have $V = V_\chi$; that is, z acts as multiplication by $\chi(z)$ on V . It is routine to verify that χ defines a \mathbb{C} -algebra homomorphism $Z_0 \longrightarrow \mathbb{C}$. \square

Let χ be as in the lemma. Define I_χ (resp. I_χ^0) to be the two-sided ideal of U_η (resp. $U_\eta(\mathfrak{g}_{\bar{0}})$) generated by the central elements

$$x_{ij} - \chi(x_{ij}), y_{ij} - \chi(y_{ij}), z_s^{\pm 1} - \chi(z_s^{\pm 1}), (i, j) \in \mathcal{I}_0, s \in [1, m+n].$$

Define the superalgebras

$$u_{\eta, \chi} =: U_\eta / I_\chi, u_{\eta, \chi}(\mathfrak{g}_{\bar{0}}) = U_\eta(\mathfrak{g}_{\bar{0}}) / I_\chi^0.$$

Lemma 7.7. $I_\chi = \mathcal{N}_{-1, \eta} I_\chi^0 \mathcal{N}_{1, \eta}$.

Proof. Since the elements $x - \chi(x)$, $x = x_{ij}, y_{ij}, z_s^{\pm 1}$ are central in U_η and all contained in $U_\eta(\mathfrak{g}_{\bar{0}})$, we have

$$\begin{aligned} I_\chi &= \sum_x U_\eta(x - \chi(x)) \\ &= \mathcal{N}_{-1, \eta} \sum_x U_\eta(\mathfrak{g}_{\bar{0}})(x - \chi(x)) \mathcal{N}_{1, \eta} \\ &= \mathcal{N}_{-1, \eta} I_\chi^0 \mathcal{N}_{1, \eta}. \end{aligned}$$

□

Corollary 7.8. *There is a \mathbb{C} -algebra isomorphism: $u_{\eta, \chi} \cong \mathcal{N}_{-1, \eta} \otimes u_{\eta, \chi}(\mathfrak{g}_{\bar{0}}) \otimes \mathcal{N}_{1, \eta}$.*

Proof. By the lemma above, we have

$$\begin{aligned} u_{\eta, \chi} &= U_\eta / I_\chi \\ &\cong \mathcal{N}_{-1, \eta} \otimes U_\eta(\mathfrak{g}_{\bar{0}}) \otimes \mathcal{N}_{-1, \eta} / \mathcal{N}_{-1, \eta} \otimes I_\chi^0 \otimes \mathcal{N}_{-1, \eta} \\ &\cong \mathcal{N}_{-1, \eta} \otimes (U_\eta(\mathfrak{g}_{\bar{0}}) / I_\chi^0) \otimes \mathcal{N}_{1, \eta} \\ &= \mathcal{N}_{-1, \eta} \otimes u_{\eta, \chi}(\mathfrak{g}_{\bar{0}}) \otimes \mathcal{N}_{1, \eta}. \end{aligned}$$

□

By Lemma 7.6, each simple U_η -module is a simple $u_{\eta, \chi}$ -module for some χ . As in [2], one can define derivations $e_{\alpha_i}, f_{\alpha_i}, k_{\pm \alpha_j}, i \in [1, m+n] \setminus m, j \in [1, m+n]$ of the superalgebra U_q by

$$e_{\alpha_i} = [E_{\alpha_i}^{(l)}, -], f_{\alpha_i} = [F_{\alpha_i}^{(l)}, -], k_{\pm \alpha_j} = [K_{\pm \alpha_j}^{(l)}, -].$$

These derivations induces derivations on U_η . By applying automorphisms of U_η as that in [1, 3.5, 3.6], [2, Th.6.1], one can assume $\chi(x_{ij}) = 0$ for any $(i, j) \in \mathcal{I}_0$ in studying simple U_η -modules or simple $U_\eta(\mathfrak{g}_{\bar{0}})$ -modules.

Assume $\chi(x_{ij}) = 0$ in the following. Denote by B_χ (resp. $B_\chi^-; U_\chi^0$) the image of B_η (resp. $B_\eta^-; U_\eta^0$) in $u_{\eta,\chi}$. Since

$$\begin{aligned} I_\chi^0 &= \sum_x U_\eta(\mathfrak{g}_{\bar{0}})(x - \chi(x)) \\ &= \left(\sum_{x=y_{ij}} B_\eta^-(x - \chi(x)) U_\eta^0 B_\eta \right. \\ &\quad \left. + B_\eta^-\left(\sum_{x=z_s^{\pm 1}} U_\eta^0(x - \chi(x)) B_\eta \right) \right. \\ &\quad \left. + B_\eta^- U_\eta^0 \left(\sum_{x=x_{ij}} B_\eta(x - \chi(x)) \right) \right). \end{aligned}$$

By a proof similar to that in Corollary 7.8, we get

$$u_{\eta,\chi} \cong B_\chi^- \otimes U_\chi^0 \otimes B_\chi.$$

In addition, B_χ is the quotient of B_η by the ideal generated by the central elements $E_{ij}^l, (i, j) \in \mathcal{I}_0$. It follows that B_χ is the algebra generated by the elements $E_{\alpha_i}, i \neq m$ and relations (R5), (R6) with q replaced by η , together with $E_{ij}^l = 0, (i, j) \in \mathcal{I}_0$.

Corollary 7.9. *The \mathbb{C} -algebra B_χ is nilpotent.*

Proof. Let G_m be the one dimensional multiplicative group ([4]). By the description of B_χ above, there is a well-defined G_m -action on B_χ defined by $t \cdot E_{ij} = t^{j-i} E_{ij}, (i, j) \in \mathcal{I}_0$. Then B_χ becomes a rational G_m -module. Since B_χ is finite dimensional, there is a largest G_m -weight $N \in \mathbb{N}$. It follows that any finite product $E_{i_1, j_1} \cdots E_{i_t, j_t} \in B_\chi$ is equal to zero, if $t > N$, since otherwise it has a G_m -weight $\sum_{s=1}^t (j_s - i_s) > N$. Thus, B_χ is nilpotent. \square

7.2 The simplicity of Kac modules for $u_{\eta,\chi}$

In this section, we study $u_{\eta,\chi}$ -modules. For the elements in U_η , we denote the images in $u_{\eta,\chi}$ by the same notation. χ is assumed to satisfy $\chi(x_{ij}) = 0$ for all $(i, j) \in \mathcal{I}_0$. Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a simple $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})\mathcal{N}_{1,\eta}$ -module. Then since $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})\mathcal{N}_{1,\eta}^+$ is a nilpotent ideal of $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})\mathcal{N}_{1,\eta}$, M is annihilated by $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})\mathcal{N}_{1,\eta}^+$. Since

$$u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})\mathcal{N}_{1,\eta}/u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})\mathcal{N}_{1,\eta}^+ \cong u_{\eta,\chi}(\mathfrak{g}_{\bar{0}}),$$

M is a simple $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})$ -module. Conversely, each $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})$ -module can be viewed as a $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})\mathcal{N}_{1,\eta}$ -module annihilated by $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})\mathcal{N}_{1,\eta}^+$.

Let M be a simple $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})$ -module annihilated by $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})\mathcal{N}_{1,\eta}^+$. Define the Kac module

$$K(M) = u_{\eta,\chi} \otimes_{u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})\mathcal{N}_{1,\eta}} M.$$

Then we have $K(M) \cong \mathcal{N}_{-1,\eta} \otimes_{\mathbb{C}} M$ as $\mathcal{N}_{-1,\eta}$ -modules.

Let $M' \subseteq M$ be a simple $U_\chi^0 B_\chi$ -submodule. Then $B_\chi M'$ is a $U_\chi^0 B_\chi$ -submodule. Since B_χ is nilpotent, $B_\chi M' = 0$, and hence M' is a simple U_χ^0 -module. Since U_χ^0 is commutative, we have that M' is 1-dimensional. Assume $M' = \mathbb{C}v$. Then there is a \mathbb{C} -algebra homomorphism λ from U_χ^0 to \mathbb{C} such that $hv = \lambda(h)v$ for all $h \in U_\chi^0$. Such an element $v \in M$ is referred to as a primitive vector of weight λ . We denote $X(U_\chi^0) = \text{Hom}_{\mathbb{C}\text{-alg}}(U_\chi^0, \mathbb{C})$.

Let M be simple $u_{\eta, \chi}(\mathfrak{g}_{\bar{0}})$ -module containing a primitive vector v_λ of weight λ . Then M is spanned by elements in the form $F_I F_0^\psi v_\lambda$ with $\psi \in [0, l)^{\mathcal{I}_0}$, $I \subseteq \mathcal{I}_1$. It follows that $M = \sum_{\mu \in X(U_\chi^0)} M_\mu$. Each $x \in M_\mu$ is called a weight vector of weight μ .

In the superalgebra $u_{\eta, \chi}$, from Sec. 5 we may assume

$$E_{\mathcal{I}_1} F_{\mathcal{I}_1} = f(K) + \sum u_i^- u_i^0 u_i^+,$$

where u_i^\pm are in the images of U_q^\pm in $u_{\eta, \chi}$, $f(K), u_i^0 \in U_\chi^0$. Then

$$E_{\mathcal{I}_1} F_{\mathcal{I}_1} v_\lambda = f(K) v_\lambda = f(K)(\lambda) v_\lambda.$$

Denote $f(K)(\lambda)$ by $f(\lambda)$.

Note that all the lemmas in Sec. 5 hold in $u_{\eta, \chi}$ (with η in place of q) as well. By a similar argument as that in Prop. 5.9, we have

Proposition 7.10. *$K(M)$ is a simple $u_{\eta, \chi}$ -module if and only if $f(\lambda) \neq 0$.*

A weight $\lambda \in X(U_\chi^0)$ is called *integral* if $\lambda(K_i^{\pm 1}) = \eta_i^{\pm \lambda_i}$ with $\lambda_1, \dots, \lambda_{m+n} \in \mathbb{Z}$. In this case, we have $\lambda = \lambda_1 \epsilon_1 + \dots + \lambda_{m+n} \epsilon_{m+n} \in \Lambda$. For each $\alpha = \epsilon_i - \epsilon_j \in \Phi^+$, set $K_\alpha = K_i K_j^{-1}$. It is then easy to check that $\lambda(K_\alpha) = \eta^{(\lambda, \alpha)}$ for any α . Moreover, for any $K^\mu, \mu \in \Lambda$, we have $\lambda(K^\mu) = \eta^{(\lambda, \mu)}$. Then by a similar argument as that for Prop. 6.2, we have $f(\lambda) = \prod_{(i, j) \in \mathcal{I}_1} [(\lambda + \rho)(K_{ij})]$, where

$$[(\lambda + \rho)(K_{ij})] = \frac{(\lambda + \rho)(K_{ij}) - (\lambda + \rho)(K_{ij}^{-1})}{\eta - \eta^{-1}}.$$

Let M be a $u_{\eta, \chi}(\mathfrak{g}_{\bar{0}})$ -module. Regard M as a $u_{\eta, \chi}(\mathfrak{g}_{\bar{0}}) \mathcal{N}_{1, \eta}$ -module annihilated by $u_{\eta, \chi}(\mathfrak{g}_{\bar{0}}) \mathcal{N}_{1, \eta}^+$. Define the induced functor from the categories of $u_{\eta, \chi}(\mathfrak{g}_{\bar{0}})$ -modules to the categories of $u_{\eta, \chi}$ -modules by

$$\text{Ind}(M) = u_{\eta, \chi} \otimes_{u_{\eta, \chi}(\mathfrak{g}_{\bar{0}}) \mathcal{N}_{1, \eta}} M.$$

Clearly Ind is an exact functor and $\text{Ind}(M) = K(M)$ in case M is a simple $u_{\eta, \chi}(\mathfrak{g}_{\bar{0}})$ -module.

For any $\mathcal{N}_{1, \eta}$ -module $N = N_{\bar{0}} \oplus N_{\bar{1}}$, denote

$$N^{\mathcal{N}_{1, \eta}^+} = \{x \in N \mid gx = 0 \text{ for any } g \in \mathcal{N}_{1, \eta}^+\}.$$

If N is a $u_{\eta, \chi}$ -module, it is easy to check that $N^{\mathcal{N}_{1, \eta}^+}$ is a (\mathbb{Z}_2 -graded) $u_{\eta, \chi}(\mathfrak{g}_{\bar{0}}) \mathcal{N}_{1, \eta}$ -submodule.

Lemma 7.11. *Let $\mathcal{N}_{1,\eta}$ be the left-regular $\mathcal{N}_{1,\eta}$ -module. Then $\mathcal{N}_{1,\eta}^{\mathcal{N}_{1,\eta}^+} = \mathbb{C}E_{\mathcal{I}_1}$.*

Proof. Using the anti-automorphism Ω , we need only show that

$$\mathcal{N}_{-1,\eta}^{\mathcal{N}_{-1,\eta}^+} = \mathbb{C}F_{\mathcal{I}_1},$$

for the right-regular $\mathcal{N}_{-1,\eta}$ -module $\mathcal{N}_{-1,\eta}$. Recall that $\mathcal{N}_{-1,\eta}$ has a basis F_I , $I \subseteq \mathcal{I}_1$. By Lemma 5.3, $F_{\mathcal{I}_1}F_{ij} = 0$ for all $(i,j) \in \mathcal{I}_1$, so that $F_{\mathcal{I}_1} \in \mathcal{N}_{-1,\eta}^{\mathcal{N}_{-1,\eta}^+}$. Let $x = \sum_{I \subseteq \mathcal{I}_1} c_I F_I \in \mathcal{N}_{-1,\eta}$. Suppose there is $I \subsetneq \mathcal{I}_1$ with $c_I \neq 0$. Let (i,j) be the largest (w.r.t the order in \mathcal{I}_1) pair not contained in some I with $c_I \neq 0$. Then by Lemma 5.3 and 5.4 we have $xF_{ij} \neq 0$. Thus $\mathcal{N}_{-1,\eta}^{\mathcal{N}_{-1,\eta}^+} = \mathbb{C}F_{\mathcal{I}_1}$. \square

Lemma 7.12. *If $\chi(z_i z_j^{-1})^2 \neq 1$ for all $(i,j) \in \mathcal{I}_1$, then $K(M)$ is simple for any simple $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})$ -module.*

Proof. Let $v_\lambda \in M$ be a primitive vector of weight λ , and let $N = N_{\bar{0}} \oplus N_{\bar{1}}$ be a nonzero submodule of $K(M)$. By a similar proof as that in Lemma 5.6 we have $F_{\mathcal{I}_1} \otimes x \in N$ for some $0 \neq x \in M$. We may assume x is a weight vector of weight μ . Since M is a simple $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})$ -module, we have $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})x = M$. Hence, there is an element

$$f = \sum c_i u_i^- u_i^0 u_i^+ \in u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})$$

such that $fx = v_\lambda$, where u_i^- (resp. u_i^+ ; u_i^0) is the product of F_{ij} (resp. E_{ij} ; $K_s^{\pm 1}$), $(i,j) \in \mathcal{I}_0$, $1 \leq s \leq m+n$, $c_i \in \mathbb{C}$.

Since x is a weight vector, we may assume $f = \sum c_i u_i^- u_i^+$. Using Lemma 5.7 and 5.8, by a minor modification of the coefficients of f , we get $f' = \sum c'_i u_i^- u_i^+$, which applied to $F_{\mathcal{I}_1} \otimes x \in N$ to get $F_{\mathcal{I}_1} \otimes v_\lambda \in N$. Applying $E_{\mathcal{I}_1}$ to which we get

$$\Pi_{(i,j) \in \mathcal{I}_1} [(\lambda + \rho)(K_{ij})] v_\lambda \in N.$$

Note that $K_{ij}^l = \chi(z_i z_j^{-1})$ in $u_{\eta,\chi}$, which implies that $[(\lambda + \rho)(K_{ij})] \neq 0$ for any $(i,j) \in \mathcal{I}_1$. Suppose otherwise $[(\lambda + \rho)(K_{ij})] = 0$ for some $(i,j) \in \mathcal{I}_1$. Then we have

$$\lambda(K_{ij}^2) = \rho(K_{ij}^{-2}) = \eta^{-2(\rho, \epsilon_i - \epsilon_j)},$$

which gives $\chi(z_i z_j^{-1})^2 = \lambda(K_{ij}^2)^l = 1$, a contradiction. Then we have $v_\lambda \in N$. Therefore $N = K(M)$, and hence $K(M)$ is simple. \square

Theorem 7.13. *If $\chi(z_i z_j^{-1})^2 \neq 1$ for all $(i,j) \in \mathcal{I}_1$, then $u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})$ and $u_{\eta,\chi}$ are Morita equivalent.*

Proof. We show that $K(M)^{\mathcal{N}_{1,\eta}^+} = M$. Note that the subspace $F_{\mathcal{I}_1} \otimes M \subseteq K(\lambda)$ is annihilated by $\mathcal{N}_{-1,\eta}^+$. Since E_{ij}, F_{ij} , $(i,j) \in \mathcal{I}_0$ commutes with $F_{\mathcal{I}_1}$ up to scalar multiple, the subspace is a simple $\mathcal{N}_{-1,\eta} u_{\eta,\chi}(\mathfrak{g}_{\bar{0}})$ -submodule of $K(M)$. Since $K(M)$ is simple, we have

$$\begin{aligned} K(M) &= u_{\eta,\chi} F_{\mathcal{I}_1} \otimes M \\ &= \mathcal{N}_{1,\eta} F_{\mathcal{I}_1} \otimes M. \end{aligned}$$

Set

$$K^-(F_{\mathcal{I}_1} \otimes M) = u_{\eta, \chi} \otimes_{\mathcal{N}_{-1, \eta} u_{\eta, \chi}(\mathfrak{g}_0)} (F_{\mathcal{I}_1} \otimes M),$$

where $F_{\mathcal{I}_1} \otimes M$ is viewed as a $\mathcal{N}_{-1, \eta} u_{\eta, \chi}(\mathfrak{g}_0)$ -module annihilated by $\mathcal{N}_{-1, \eta}^+ u_{\eta, \chi}(\mathfrak{g}_0)$. By the comparison of dimensions we have that $K(M)$ is isomorphic to $K^-(F_{\mathcal{I}_1} \otimes M)$ as $u_{\eta, \chi}$ -modules. Thus, as $\mathcal{N}_{1, \eta}$ -modules, we have

$$K(M) \cong \mathcal{N}_{1, \eta} \otimes_{\mathbb{F}} F_{\mathcal{I}_1} \otimes M,$$

from which it follows that

$$\begin{aligned} K(M)^{\mathcal{N}_{1, \eta}^+} &\cong (\mathcal{N}_{1, \eta})^{\mathcal{N}_{1, \eta}^+} \otimes F_{\mathcal{I}_1} \otimes M \\ &\cong E_{\mathcal{I}_1} F_{\mathcal{I}_1} \otimes M \\ &= M, \end{aligned}$$

where the last equality is given by the fact that $E_{\mathcal{I}_1} E_{\mathcal{I}_1} v_\lambda \neq 0$.

From above discussion, we have that the functor $(\cdot)^{\mathcal{N}_{1, \eta}^+}$ is right adjoint to Ind. By a similar argument as that for [3, Th. 3.2], $u_{\eta, \chi}(\mathfrak{g}_0)$ and $u_{\eta, \chi}$ are Morita equivalent. \square

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